International Journal of Engineering Research & Informatics (IJERI) ISSN: 2348-6481
Vol.5, Issue No 1, 2025

Two-phase nonlinear inverse Stefan problem in one dimension: a meshless approach

Mr. Khasim Pasha Syed, Asst Prof, Dept. of Maths, SBIT. jccpasha748@gmail.com, 9849243151

Mr. N Koteswar Rao, Asst Prof, Dept. of Maths, SBIT. kotichowdaryn@gmail.com, 9703299161

Ms. P Himabindu, Asst Prof, Dept. of Maths, SBIT. bindu68287@gmail.com, 9440768287

Abstract :

For the one-dimensional, two-phase, inverse linear Stefan problem, we expand a meshless approach of basic solutions recently proposed by the authors to the nonlinear case. This latter scenario, which is more realistic from a practical standpoint, likewise treats the free surface as uncertain. A linear combination of the basic solutions to the heat equation is used to estimate the solution in each phase, building on past research. Since one must deal with a nonlinear minimization issue in the current scenario in order to locate the free surface, implementation and analysis are more challenging. Additionally, the inverse problem is poorly posed since even tiny inaccuracies in the measured input data might result in significant variations from the ideal outcome.Consequently, regularization must be included in the target function that is reduced in order to arrive at a reliable answer. Results from calculations are shown and discussed.

1. Introduction

The two-phase direct Stefan problem requires determining the temperature distribution and the moving free interface when the initial and boundary conditions, as well as the thermal properties of the bi-material involved, are known, see e.g. [19]. Under various boundary conditions it was shown to be well-posed, see [3,5,6]. In contrast to the direct problem, inverse Stefan problems require determining some initial temperature and/or boundary conditions, and/or thermal properties from additional information, which may involve the partial knowledge of the free surface, the temperature measured at some points inside the medium or on the boundary, the heat flux, etc., see [10]. In comparison to the studies on the one-phase flow, the literature on solving two-phase inverse Stefan problems is much more scarce, see [1,17,20]. However, these inverse design Stefan problems were linear because in theirformulation the position of the moving interface is considered known. When the position of the moving interface is unknown and also no temperature or heat flux boundary conditions are specified on a part of the boundary then one deals with a nonlinear and ill-posed inverse Stefan problem, see [11]. Although the uniqueness of a solution in Hölder spaces for such a class of two-phase inverse nonlinear Stefan problems holds, see [10,12], these problems are still ill-posed because there is no continuous dependence of the solution on the input data. The plan of the paper is as follows. In Section 2, we give the mathematical formulation of the one-dimensional two-phase inverse nonlinear Stefan problem and point out its ill-posedness. In Section 3, we describe the regularized numerical meshless method of fundamental solutions (MFS) for constructing a stable solution to the inverse problem. Section 4 presents and discusses numerical results obtained for some typical test examples with and without noise included in the input data. Finally, Section 5 presents conclusions and possible future work.

2. Mathematical formulation

Assume that the interface

$$s(t) \in (0, l], \quad \text{for } t \in (0, T], \tag{2.1}$$

and $s(0)$ is given, and denote by

$$Q_T = \{(x, t) \in (0, l] \times (0, T]\}$$

the two-phase rectangular domain $(0, l) \times (0, T]$, which is subdivided by the interface into the two subdomains

$$D_T^1 = \{(x, t) \in Q_T | 0 < x < s(t), \quad t \in (0, T]\},$$

$$D_T^2 = \{(x, t) \in Q_T | s(t) < x < l, \quad t \in (0, T]\}.$$

We investigate the inverse nonlinear two-phase one-dimensional Stefan problem which requires finding the triplet solution $(u_1, u_2, s) \in C^{2,1}(D_T^1) \times C^{2,1}(D_T^2) \times (C([0, T]) \cap C^1(0, T])$, satisfying (2.1), the heat equations

$$\frac{\partial u_1}{\partial t} = \alpha_1 \frac{\partial^2 u_1}{\partial x^2}, \quad (x, t) \in D_T^1 \tag{2.2a}$$

$$\frac{\partial u_2}{\partial t} = \alpha_2 \frac{\partial^2 u_2}{\partial x^2}, \quad (x, t) \in D_T^2 \tag{2.2b}$$

where $\alpha_n > 0$ is the thermal diffusivity of the heat conductor D_T^n for $n = 1, 2$, the initial conditions at $t = 0$

$$u_1(x, 0) = u_1^0(x), \quad x \in [0, s(0)], \tag{2.3a}$$

$$u_2(x, 0) = u_2^0(x), \quad x \in [s(0), l], \tag{2.3b}$$

the interface Stefan conditions at $x = s(t)$

$$u_1(s(t), t) = u_2(s(t), t) = u^*(t), \quad t \in (0, T], \tag{2.4a}$$

$$s'(t) = -K_1 \frac{\partial u_1}{\partial x}(s(t), t) + K_2 \frac{\partial u_2}{\partial x}(s(t), t), \quad t \in (0, T], \tag{2.4b}$$

where $K_n = k_n/(\rho_2 L)$, and k_n, ρ_n, L are the thermal conductivities, densities and latent heat, respectively, of D_T^1 (water) and D_T^2 (ice), and the Cauchy boundary conditions at $x = l$

$$u_2(l, t) = f_l(t), \quad t \in (0, T], \tag{2.5a}$$

$$k_2 \frac{\partial u_2}{\partial x}(l, t) = g_l(t), \quad t \in (0, T], \tag{2.5b}$$

We also impose the compatibility conditions at the corners $(x, t) = (s(0), 0)$ and $(l, 0)$, namely

$$u_1^0(s(0)) = u_2^0(s(0)) = u^*(0), \tag{2.6a}$$

$$u_2^0(l) = f_l(0), \quad k_2 \frac{du_2^0}{dx}(0) = g_l(0). \tag{2.6b}$$

B.T. Johansson et al. / Mathematics and Computers in Simulation 101 (2014) 61–77

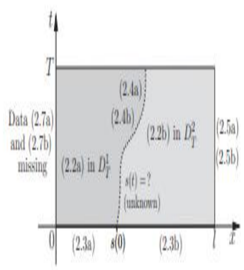

Fig. 1. Sketch of the two-phase inverse nonlinear Stefan problem (2.1)–(2.5).

Remark that the fixed boundary $x = l$ is overspecified because both Dirichlet and Neumann (i.e. Cauc
prescribed in (2.5a) and (2.5b), whilst the fixed boundary $x = 0$ is underspecified because no condition is pı
it. Physical quantities of interest related to this hostile or inaccessible boundary are the temperature

$$u_1(0,t) = f_0(t), \quad t \in (0,T],$$

the heat flux

$$-k_1 \frac{\partial u_1}{\partial x}(0,t) = g_0(t), \quad t \in (0,T],$$

and the mass

$$\int_0^{s(t)} u_1(x,t)\,dx = E(t), \quad t \in (0,T].$$

This data can also be required to satisfy the compatibility conditions at the origin $(x,t) = (0,0)$, namely

$$f_0(0) = u_1^0(0), \quad g_0(0) = -k_1 \frac{du_1^0}{dx}(0), \quad E(0) = \int_0^{s(0)} u_1^0(x)\,dx.$$

A sketch of the conditions in the two-phase inverse nonlinear Stefan problem is shown in Fig. 1.

Sometimes, in practice, the measurement of both the boundary temperature (2.5a) and the heat flux (2.
be easy and only one of this data may be available. In such a situation one could measure instead the upp
final time) internal temperature at $t = T$, namely

$$u_1(x,T) = u_1^T(x), \quad x \in [0, s(T)],$$

$$u_2(x,T) = u_2^T(x), \quad x \in [s(T), l],$$

where $s(T) \in (0,l)$ is also given. However, it turns out, see [11], that this latter 'upper base data (2.9)' inv
is more ill-posed than the former 'Cauchy data (2.5a and 2.5b)' inverse problem because it can have m
solution, i.e. the solution is not unique. However, as we shall see below, the former problem can have
solution, i.e. the solution is unique, though the problem is still ill-posed since small errors in the input d
cause large errors in the output data (2.7). Therefore, in this study only the inverse problem given by Eqs
will be investigated. This problem may be considered as a continuation problem of the solution of the pa
heat equation from the boundary $x = l$, where the Cauchy data (2.5a and 2.5b) is given, into the domain Q
be reinterpreted as a "backward in space" inverse heat conduction problem with an "initial" transient b
[18]. However, in contrast to the non-characteristic Cauchy problem, there is the unknown phase transi
boundary $x = s(t)$ in Q_T also to be determined and this essentially complicates the task of analytic contin

An initial attempt would be to split the two-phase inverse Stefan problem (2.1)–(2.6) into two proble
is the nonlinear inverse boundary determination problem for the pair (u_2, s) satisfying equations (2.1), (
(2.5a and 2.5b), (2.6) and

$$u_2(s(t),t) = u^*(t), \quad t \in (0,T]$$

with

$$u_2^0(s(0)) = u^*(0). \tag{2.11}$$

The solution of this problem is unique, see [9], even if the initial condition (2.3b) is not imposed, see [22]. Once the boundary $x = s(t)$ and the heat flux $\frac{\partial u_2}{\partial x}(s(t),t)$ have been determined, the second is the linear inverse problem for determining the temperature u_1 satisfying equations (2.2a), (2.3a),

$$u_1(s(t),t) = u^*(t), \quad t \in (0,T] \tag{2.12}$$

with

$$u_1^0(s(0)) = u^*(0), \tag{2.13}$$

and

$$-K_1 \frac{\partial u_1}{\partial x}(s(t),t) = s'(t) - K_2 \frac{\partial u_2}{\partial x}(s(t),t), \quad t \in (0,T]. \tag{2.14}$$

The solution of this problem is unique, see [4] even if the initial condition (2.3a) is not imposed, see [14,21]. However, both the above problems are ill-posed since their solutions do not depend continuously on the input data. In order to obtain stable solutions regularization is necessary and this involves choosing at least two regularization parameters, one for each problem. Besides depending on the amount of noise in the input data they depend on each other since the first nonlinear ill-posed problem has to be solved first to provide the input for the second linear ill-posed problem. So, this two-parameter choice becomes complicated.

Therefore, it appears more useful to solve the composite problem (2.1)–(2.6) in one go for simultaneously determining the solution (u_1, u_2, s).

The problem is still ill-posed but it can now be regularized by choosing a single regularization parameter for the unknown temperature fields. This combined approach has been used previously by the authors for simultaneously determining a heat source and the initial temperature, see [15]. Finally, we will also investigate the case when the initial conditions (2.3a) and (2.3b), together with the compatibility conditions (2.6a) and (2.6b), are not prescribed.

3. The method of fundamental solutions

We approximate the solutions (u_1, u_2) of the heat equations (2.2a) and (2.2b) using the method of fundamental solutions (MFS) for the unsteady heat conduction in composite layered materials, recently developed by the authors in [7,16,17]. In the MFS, we approximate u_1 and u_2 by linear combinations of fundamental solutions of the heat equation

$$G_n(x,t;y,\tau) = \frac{H(t-\tau)}{\sqrt{4a_n\pi(t-\tau)}} \exp\left(-\frac{(x-y)^2}{4a_n(t-\tau)}\right), \quad n = 1,2, \tag{3.1}$$

where H is the Heaviside function, of the form

$$u_n(x,t) = \sum_{j=1}^{M} c_j^{(n)} G_n(x,t;y_j^{(n)},\tau_j), \quad (x,t) \in \overline{D_T^n}, \; n = 1,2. \tag{3.2}$$

In expression (3.2), the source points $(y_j^{(n)}, \tau_j)$ for $j = \overline{1,M}$ and $n = 1,2$, are located outside the solution domains $\overline{D_T^n}$ in the following way:

$$y_j^{(1)} = \begin{cases} -h, & j = \overline{1, 2M_1} \\ h + s_{3M_1-j+1}, & j = \overline{2M_1+1, 3M_1} \\ h + s_{4M_1-j+1}, & j = \overline{3M_1+1, 4M_1} \end{cases} \tag{3.3a}$$

$$y_j^{(2)} = \begin{cases} -h + s_{M_1-j+1}, & j = \overline{1, M_1} \\ -h + s_{2M_1-j+1}, & j = \overline{M_1+1, 2M_1} \\ l + h, & j = \overline{2M_1+1, 4M_1} \end{cases} \tag{3.3b}$$

B.T. Johansson et al. / Mathematics and Computers in Simulation 101 (2014) 61–77

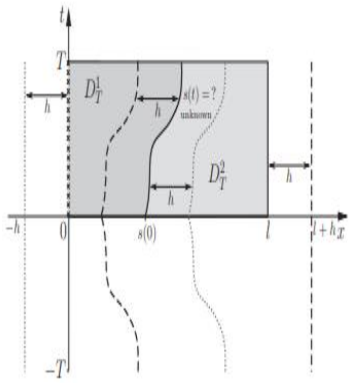

Fig. 2. Representation of the two-phase problem with domains D_T^1 and D_T^2, unknown boundary data (×), and source points (•••) and (- externally to the domains D_T^1 and D_T^2, respectively.

where h is a preassigned MFS positive parameter giving the distance between the source points and the bo $M = 4M_1$,

$$\tau_j = \frac{(2j-1-2M_1)T}{2M_1} \quad \text{for} \quad j = \overline{1, 2M_1},$$

$\tau_j = \tau_{j-2M_1}$ for $j = \overline{2M_1+1, 4M_1}$, and $s_j = s(\tau_{j+M_1})$ for $j = \overline{1, M_1}$, see Fig. 2 for a representation of the d boundaries and placement of source points.

The $2M$ unknown coefficients $\mathbf{c} = \left(c_j^{(n)}\right)_{j=\overline{1,M}}^{n=1,2}$ and the $M_1 = M/4$ discretization points $\mathbf{s} = (s_j)_{j=\overline{1,M_1}}$ of the u free boundary have to be determined by collocating the conditions (2.3)–(2.6), as described next. Let us select a distribution of collocation points given by

$$t_0 = 0, \quad t_i = \frac{(2i-1)T}{2M_1} = |\tau_{2M_1-i+1}|, \quad i = \overline{1, M_1},$$

$$\tilde{t}_i = \frac{iT}{M_2} \quad \text{for } i = \overline{0, M_2},$$

$$x_1^{(k)} = \frac{ks(0)}{K+1}, \quad x_2^{(k)} = s(0) + \frac{k(l-s(0))}{K+1}, \quad k = \overline{1, K},$$

and collocate equations (2.3)–(2.6) as follows:

$$u_n(x_n^{(k)}, 0) = u_n^0(x_n^{(k)}), \quad k = \overline{1, K}, \; n = 1, 2,$$

$$u_1(s_i, t_i) = u_2(s_i, t_i) = u^*(t_i), \quad i = \overline{0, M_1},$$

$$-K_1 \frac{\partial u_1}{\partial x}(s_i, t_i) + K_2 \frac{\partial u_2}{\partial x}(s_i, t_i) = s'(t_i) = \begin{cases} \dfrac{s(t_1) - s(0)}{t_1}, & i = 1 \\ \dfrac{s(t_i) - s(t_{i-1})}{t_i - t_{i-1}}, & i = \overline{2, M_1} \end{cases}, \quad i = \overline{1, M_1},$$

$$u_2(l, \tilde{t}_i) = f_l(\tilde{t}_i), \quad k_2 \frac{\partial u_2}{\partial x}(l, \tilde{t}_i) = g_l(\tilde{t}_i), \quad i = \overline{0, M_2}.$$

In total, via (3.2), Eqs. (3.5)–(3.8) form a system of $(2K+3M_1+2M_2+4)$ equations with $9M_1$ unknowns (general, we require to have at least as many equations as unknowns and therefore we require

$$2M_2 + 2K + 4 \geq 6M_1, \quad \text{or} \quad M_2 + K + 2 \geq 3M_1.$$

Note that this system is linear in $\mathbf{c}$, but it is nonlinear in $\mathbf{s}$. In addition, our inverse problem is ill-posed. Therefore, we apply the nonlinear Tikhonov regularization method based on minimizing the non-linear regularized least-squares functional

$$\begin{aligned} \mathcal{F}(\mathbf{c}, \mathbf{s}) = {} & \sum_{n=1}^{2}\sum_{k=1}^{K}\left(\sum_{j=1}^{M} c_j^{(n)} G_n(x_n^{(k)}, 0; y_j^{(n)}, \tau_j) - u_n^0(x_n^{(k)})\right)^2 + \sum_{i=0}^{M_1}\left(\sum_{j=1}^{M} c_j^{(1)} G_1(s_i, t_i; y_j^{(1)}, \tau_j) - u^*(t_i)\right)^2 \\ & + \sum_{i=0}^{M_1}\left(\sum_{j=1}^{M} c_j^{(2)} G_2(s_i, t_i; y_j^{(2)}, \tau_j) - u^*(t_i)\right)^2 \\ & + \sum_{i=1}^{M_1}\left(\sum_{j=1}^{M}\left[K_2 c_j^{(2)} \frac{\partial G_2}{\partial x}(s_i, t_i; y_j^{(2)}, \tau_j) - K_1 c_j^{(1)} \frac{\partial G_1}{\partial x}(s_i, t_i; y_j^{(1)}, \tau_j)\right] - s'(t_i)\right)^2 \\ & + \sum_{i=0}^{M_2}\left(\sum_{j=1}^{M} c_j^{(2)} G_2(l, \tilde{t}_i; y_j^{(2)}, \tau_j) - f_l(\tilde{t}_i)\right)^2 + \sum_{i=0}^{M_2}\left(\sum_{j=1}^{M} c_j^{(2)} k_2 \frac{\partial G_2}{\partial x}(l, \tilde{t}_i; y_j^{(2)}, \tau_j) - g_l(\tilde{t}_i)\right)^2 \\ & + \lambda_1 |\mathbf{c}|^2 + \lambda_2 |\mathbf{s}|^2, \end{aligned} \tag{3.9}$$

where $\lambda_1, \lambda_2 \geq 0$ are regularization parameters which can be prescribed according to some criterion, e.g. the L-surface, or, more simply, by trial and error. Note that in the last term of (3.9), for simplicity and in a first attempt, we have imposed that $s \in C[0, T]$, but more regularity such as $s \in C^1[0, T]$ can also be imposed in order to get some stability estimates.

In imposing the flux boundary conditions (2.4b) and (2.5b) the x-partial derivative of (3.1) is needed, as given by

$$\frac{\partial G_n}{\partial x}(x, t; y, \tau) = -\frac{(x-y)H(t-\tau)}{2\sqrt{4\pi\alpha_n^3(t-\tau)^3}} \exp\left(-\frac{(x-y)^2}{4\alpha_n(t-\tau)}\right), \quad n = 1, 2.$$

The minimization of the functional (3.9) is performed using the MATLAB toolbox *lsqnonlin* which is designed to minimize a sum of squares of arbitrary differentiable functions. The gradient does not need to be supplied by the user and the simple bounds on the variables

$$0 < s_i < l \quad \text{for } i = \overline{1, M_1} \tag{3.10}$$

are also allowed.

The initial guess to start the iterative process is arbitrary and in this study we take $\mathbf{c}^0 = \mathbf{0}$ and $\mathbf{s}^0 = \mathbf{s}(0)$.

4. Numerical results and discussion

In this section, we apply the MFS outlined in the previous section to the inverse two-phase nonlinear Stefan problem (2.1)–(2.6). In the first two examples we have analytical solutions available and we also investigate for one of them the case when the initial conditions (2.3a) and (2.3b) are not given. In the third example, an analytical solution is not available. Previous experience with applying the MFS for the heat equation, [16,17], suggests that the MFS parameter h should be not too small (which will result in less accurate approximations) nor too large (which will increase the ill-conditioning of the system). In this section, the value of h, as well as the values of the regularization parameters λ_1 and λ_2, are chosen by trial and error. Nevertheless, more rigorous investigations on these choices should be undertaken in a future work.

Conclusions

This work uses a regularized MFS to investigate the one-dimensional two-phase nonlinear inverse Stefan issue. The approach has been demonstrated to be accurate, consistent, and resilient for both precise and noisy data. Since this study conducts the first numerical inquiry to resolve the inverse two-phase nonlinear Stefan problem, there are currently no results to compare our MFS with. We point out that an alternative would be to use the boundary The problem can be solved using the element method, a potent and effective numerical boundary discretization technique.The MFS established in this study will be expanded in future work to multi-dimensional nonlinear Stefan issues.

Acknowledgements:

T. Reeve expresses gratitude for the EPSRC's financial assistance. In order to complete some of this work, B.T. Johansson and D. Lesnic would like to thank the Isaac Newton Institute for the Visiting Fellowships provided as part of the program on "Inverse Problems" presented in November-December 2011. The remarks andThe referees' suggestions are acknowledged.

References

[1] D.D. Ang, A. Pham Ngoc Dinh, D.N. Tranh, Regularization of an inverse two-phase Stefan problem, Nonlinear Anal. 34 (1998) 719–731.

[2] J.V. Beck, B. Blackwell, C.R. St-Clair Jr., Inverse Heat Conduction, J. Wiley-Intersc. Publ., New York, 1985.

[3] J.R. Cannon, D.B. Henry, D.B. Kotlow, Classical solutions of the one-dimensional, two-phase Stefan problem, Ann. Mat.Pura Appl. 107 (1975) 311–341.

[4] J.R. Cannon, J. Douglas Jr., The Cauchy problem for the heat equation, SIAM J. Numer. Anal. 4 (3) (1967) 317–336.

[5] J.R. Cannon, M. Primicerio, A two phase Stefan problem with flux boundary conditions, Ann. Mat. Pura Appl. 88 (1971) 193–205.

[6] J.R. Cannon, M. Primicerio, A two phase Stefan problem with temperature boundary conditions, Ann. Mat. Pura Appl. 88 (1971) 177–191.

[7] S. Chantasiriwan, B.T. Johansson, D. Lesnic, The method of fundamental solutions for free surface Stefan problems, Eng. Anal.Bound. Elem. 33 (2009) 529–538.

[8] J. Crank, Free and Moving Boundary Problems, Clarendon Press, Oxford, 1984.

[9] A. El Badia, F. Moutazaim, A one-phase inverse Stefan problem, Inverse Prob. 15 (1999) 1507–1522.

[10] N.L. Gol'dman, Inverse Stefan Problems, Kluwer Academic Publ., Dordrecht, 1997.

[11] N.L. Gol'dman, Inverse problems with phase transitions., in: O.M. Alifanov, N.V. Kerov, V.V. Michailov, A.N. Nenarokomov (Eds.), Proc. 3rd Int. Conf. Dynamic Systems Identification and Inverse Problems, Moscow-St. Petersburg, Russia, 1998, pp. 128–139.

[12] N.L. Gol'dman, Properties of solutions of the inverse Stefan problem, Differ. Equ. 39 (2003) 66–72.

[13] D. NhoHào, Methods for Inverse Heat Conduction Problems, Peter Lang, Frankfurt am Main, 1998.

[14] C.D. Hill, Parabolic equations in one space variable and the non-characteristic Cauchy problem, Comm. Pure Appl. Math. 20 (1967) 619–633.

[15] B.T. Johansson, D. Lesnic, A procedure for determining a spacewise dependent heat source and the initial temperature, Appl. Anal. 87 (2008) 265–276.

[16] B.T. Johansson, D. Lesnic, A method of fundamental solutions for transient heat

conduction in layered materials, Eng. Anal. Bound. Elem. 33 (2009) 1362–1367.

[17] B.T. Johansson, D. Lesnic, T. Reeve, A meshless method for an inverse two-phase one-dimensional linear Stefan problem, Inverse Prob. Sci. Eng. 21 (1) (2013) 17–33.

[18] D.A. Murio, Solution of inverse heat conduction problems with phase changes by the mollification method, Comput. Math. Appl. 24 (1992) 45–57.

[19] L. Rubinsteın, The Stefan problem: Comments on its present state, J. Inst. Maths Appl. 24 (1979) 259–277.

[20] D. Slota, Solving the inverse Stefan design problem using genetic algorithms, Inverse Prob. Sci. Eng. 16 (2008) 829–846.

[21] Y.B. Wang, J. Cheng, J. Nakagawa, M. Yamamoto, A numerical method for solving the inverse heat conduction problem without the initial value, Inverse Prob. Sci. Eng. 18 (2010) 655–671. [22] T. Wei, M. Yamamoto, Reconstruction of a moving boundary from Cauchy data in one-dimensional heat equation, Inverse Prob. Sci. Eng. 17 (2009) 551–567